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Analysis of the Interaction between A Fluid and A Circular Pile Using the Fractional Navier-Stokes Equations

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ABSTRACT
The purpose of this research is to study the interaction between a fluid and a circular pile, located downstream from a fan-shaped dam, through the fractional Navier-Stokes equations, and in particular, its approximation to the boundary layer. The flow region is divided into zones according to the vorticity transport theory of turbulence. First, we consider the limit of the spatial occupancy index close to 1. Then, a stream function is introduced, and for the potential zone, we consider a complex potential, using the inverse distances on a circle. In the other limit, when the spatial occupation index approaches 0, we consider the equations of the boundary layer in the limit of fully developed turbulence. Next, for the last approaches, a new stream function and velocities in their radial and polar components are obtained. We also find the asymmetry of the pressure distribution around the pile, based on the viscosity and considering that the pressure drag force and the friction coefficients are proportional to the inverse of the Reynolds number. We conclude that D'Alembert's paradox and Thomson's theorem has been resolved. For applications, in the case of the turbulent wake, we are interested both in the orientation given by the pile symmetry axis and its extension. The criterion that should be satisfied is: the diameter of the pile, on the border of inequality, must be located as proportional average between the length of the turbulent wake and twice the characteristic length associated with the dam, whose aspect ratio, in turn, to the pile diameter, determines the contraction factor.
***Keywords* –** Fractional Navier-Stokes equations, dam, circular pile, D'Alembert's paradox, Thomson's theorem

I. INTRODUCTION

The main objective of this research is to study the interaction between a fluid and a pile, particularly in the leeward region of the pile. In addition to restoring coherence, we differentiate between highly turbulent limit and inviscid fluid, thus resolving D'Alembert paradox and reformulating Thomson's theorem on circulation. Here we use the Riemann-Liouville fractional derivative concept.

We approach the case of a cylindrical pile with circular cross section, smooth surface, oriented transversely-vertically to the main flow direction, and fixed on a rigid bed.

The fluid movement is described by means of the fractional Navier-Stokes equations, and particularly by its approximation to the boundary layer. Within the Eulerian formulation as the velocity vector field and the Helmholtz theorem, the vector field is recovered from two potential types: vector and scalar. So the field is divided into zones of nonzero vorticity and zero vorticity or potential zone according to the scalar field theory, being the rotational the smaller area, corresponding to experimental results, as also occurs especially in the pile lee side.

In the windward side of the pile, due to an inverse pressure gradient, a lifting of the boundary layer from the bed and a winding in the downstream direction occurs, which causes the knot vortex (or horseshoe shape) that is set around the pile, and its reproduction by reiteration [1].

First, we consider the limit of the spatial occupation index near 1. To incorporate the pile shape, we describe the advection acceleration in curvilinear coordinates and obtain the solution by applying the boundary layer equations in cylindrical coordinates.

The statement of a stream function reduces the three governing equations to only two. For the potential area, we consider the complex potential, whose real part is the potential (scalar) and its imaginary part is the stream function; then we find the velocities and pressures. In the other limit, the spatial occupation index approaches 0; we consider the boundary layer equations in the limit of fully developed turbulence, then a new stream function and velocities in their radial and polar components are obtained.

In the applications field, the case of dams, fan-shaped, the interaction between the fluid and a cylindrical pile is present, which must be located in a large area of flow control, in order to guide the turbulent wake and to reduce its energy before the arrival to the control section and the transition zone, and the beginning of the discharge channel. So, we are interested in the orientation and length of the turbulent wake downwind of the pile. The first is set by the symmetry axis of the pile according to the current, while the length is imposed as a restriction to suggest a level for the aspect ratio of the same with the pile diameter.

II. PROBLEM FORMULATION AND SOLUTION OF THE MODEL

The law that governs fluid motion is described by the fractional Navier-Stokes equations together with the mass conservation law, which is reduced to the condition of non-divergence, in the case of incompressible fluids [2], [3], [4]. We observe an inability of the classic version of the Navier Stokes equations to explain some hydraulics problems like: the interaction between the fluid and a flat surface; the inverse cascade and vortices formation; the water profile in a spillway ogee shaped crest and the closure problem in Reynolds equations.

According to our description, fluid movement occurs on the basis of viscous forces between adjacent layers with different velocities; therefore it cannot be described by a local operator due to the participation of a frictional force, and must be expressed by means a non-local operator. This is achieved by the law where Darcy's flow is proportional and opposite to the fractional gradient of momentum per unit volume. Darcy's flow generates an exchange of momentum, so that according to Newton's law, momentum change is the negative divergence, or Darcy's flow convergence, with opposite sign. Variations in pressure and body forces also contribute to momentum change. Finally, for an incompressible fluid we obtain its velocity field evolution.

Momentum Darcy's flow is described by the equation

$$\mathbf{q}_D = -\nu_\alpha \nabla_M^\beta \rho \mathbf{u}, \qquad (1)$$

where $\rho\mathbf{u}$ is the momentum per unit volume, $\nabla_M^\beta \rho\mathbf{u}$ is the fractional gradient, ν_α, $\alpha = 1 + \beta$ is the momentum diffusivity or kinematic α-viscosity, β is the spatial occupation index, with the following units ($[\nu_\alpha] = m^\alpha / \sec$), and M is the scale mixture for different spatial directions.

Taken into account the different momentum changes and denoting by (p, ϕ, ρ) the pressure, the potential of the body force and the fluid density, we obtain that the derivative of the velocity $\mathbf{u}$ is given by

$$\frac{\partial}{\partial t}\mathbf{u} = -\nu_\alpha (-\Delta)^{\alpha/2}\mathbf{u} + \mathbf{u} \times rot\,\mathbf{u} - \nabla\left(\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}) + \frac{p}{\rho} + \phi\right), \qquad (2)$$

Where M is chosen so that the flow $\nu_\alpha \nabla_M^\alpha \mathbf{u}$ be proportional to the negative of the fractional Laplacian $-\nu_\alpha(-\Delta)^{\alpha/2}\mathbf{u}$. Thus, equation (2) indicates that vorticity energizes the velocity field change against viscosity and under the restriction of energy conservation [4], [5].

The velocity and pressure changes could be understand as deviations with respect to its average time value, and result in random stationary variables with zero-mean. However, we denoted mean values by (u_i, p) [5]. The viscous flow in equation (2) can be rewritten in the form $\nu_\alpha \nabla_M^\alpha \mathbf{u}$, and based on the ergodic assumption, the Reynolds equations can be obtained, which contain the Reynolds stresses ($\tau_{ij} = -\rho\left\langle \delta u_i \delta u_j \right\rangle$) and the deformation index rate ($S_{ij\beta} = \frac{1}{2}\left(\frac{\partial^\beta u_i}{\partial x^{j\beta}} + \frac{\partial^\beta u_j}{\partial x^{i\beta}}\right)$). Then equation (2) can be written in the form:

$$\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x^j}\left\langle u_i u_j \right\rangle = -\frac{1}{\rho}\frac{\partial}{\partial x^i}p + \frac{\partial}{\partial x^j}\left(2\nu_\alpha S_{ij\beta} + \frac{1}{\rho}\tau_{ij}\right) \qquad (3)$$

Boundary layer equations could be obtained by means of the fractional Navier-Stokes equations taken into consideration its relatively thin thickness, and are shown below $\partial_t u + u\partial_x u + v\partial_y u = -\partial_x p/\rho + \nu_\alpha \partial_y^\alpha u$, [6], [7]. Also the stationary version of the boundary layer equations and the mass conservation equation as null divergence are shown in (4):

$$u\partial_x u + v\partial_y u = -\partial_x p/\rho + \nu_\alpha \partial_y^\alpha u, \quad \partial_x u + \partial_y v = 0 \qquad (4)$$

Cartesian boundary layer equations take the form shown in (5), where the highly turbulent boundary is represented by $\beta \to 0$, and $\nu_1 = \nu_c / l$,

$$u\partial_x u + (\tilde{v})\partial_y u = -\partial_x p/\rho, \qquad \tilde{v} = v - \nu_c / l$$
$$\partial_x u + \partial_y(\tilde{v}) = 0, \qquad \tilde{v}\big|_{r=a} = -\nu_c / l$$

III. SOLUTIONS

In order to adapt to the different forms we use curvilinear coordinates, and modify the advective derivative $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left((1/2)u^2\right) - \mathbf{u} \times rot\,\mathbf{u}$. In cylindrical coordinates and in the approximation of the boundary layer, the concentrated expression of the advective derivative is simplified as follows: $\left((\mathbf{u}\cdot\nabla)u_r - (1/r)u_\theta^2, (\mathbf{u}\cdot\nabla)u_\theta + (1/r)u_r u_\theta\right)$, considering also that

$$\nabla_{pol} = (\partial_r, (1/r)\partial_\theta), \quad (u_r(r,\theta), u_\theta(r,\theta))$$

, equation that replace the Cartesian form $\nabla_{car} = (\partial_x, \partial_y), \quad (u_x, u_y)$. Therefore, the equation becomes as shown in (6), wherein, given its importance, polar component is first written,

$$u_r\partial_r u_\theta + u_\theta(1/r)\partial_\theta u_\theta + (1/r)u_r u_\theta = -\partial_\theta p/\rho + \nu_\alpha \partial^\alpha u_\theta$$
$$u_r\partial_r u_r + u_\theta(1/r)\partial_\theta u_r - (1/r)u_\theta^2 = -\partial_r p/\rho + \nu_\alpha \partial^\alpha u_r$$
$$\partial_r u_r + (1/r)\partial_\theta u_\theta = 0 \tag{6}$$

The coordinate's origin is located in the center of the pile whose radio is a, and on its surface is placed the boundary condition $\mathbf{u}|_{r=a} = 0$.

Mass conservation is expressed as the coincidence of the mixed derivatives which manifests the continuity of the partial derivatives of the stream function. Now the three equations system is reduced to two, the polar (7) and the radial (8), being ν_c the cinematic viscosity,

$$(\partial_r\psi)\partial_\theta(\partial_r\psi) - \frac{\partial_\theta\psi\,\partial_r\psi}{r} =$$
$$-\partial_\theta p/\rho + \nu_c\left(-\frac{\partial_\theta^2(\partial_r\psi)}{r} + \frac{\partial_r\psi}{r} + \frac{2\partial_\theta^2\psi}{r^2}\right)$$

(7)

$$(-\partial_r\psi)\frac{\partial_\theta^2\psi}{r^2} - \frac{(\partial_r\psi)^2}{r} =$$
$$-\partial_r p/\rho + \nu_c\left(\frac{\partial_\theta^3\psi}{r^3} - \frac{\partial_\theta\psi}{r^3} + \frac{2\partial_\theta\partial_r\psi}{r^2}\right)$$

(8)

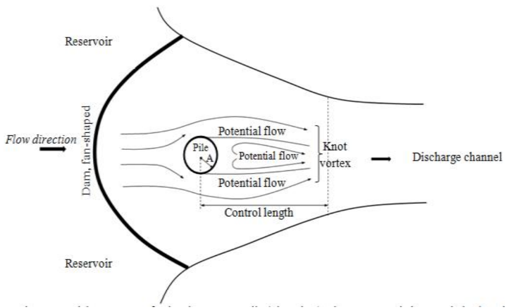

Fig. 1. Flow control downstream of a dam by means a pile (plan view). Flow zones and characteristics lengths are shown.

The pile shadow is divided in the following two flow zones: irrotationals (or potential), where velocity is given by a scalar potential and rotationals (or solenoidal), where the vorticity is nonzero. Both are shown in Figure (1), [8]. For the potential or irrotational zone, the complex potential is formed by means of the scalar potential and the stream function $f(z) = w = \phi + i\psi$, and is obtained from the inverse of $w = 1/z = \bar{z}/|z|^2$. But in geometrical way it is described as $|w||z| = 1$, so the circle radius is the result of multiplying the outer distance by the inland. Changing the location of the circle center and its radius we obtained that the circle radius remains as the proportional average of the following two distances $\frac{|w - z_0|}{a\sqrt{U_l}} = \frac{a\sqrt{U_l}}{|z - z_0|}$, while its product is proportional to the flow that would pass through a tube with similar cross section.

Besides, $a\sqrt{U_l}$ is the proportional average between the outer and inland distances [9]. Resolving we obtained (9)

$$\psi = \frac{a^2 U_l}{r}\sin\theta, \quad \phi = -\frac{a^2 U_l}{r}\cos\theta,$$
$$f(z) = -\frac{a^2 U_l}{z} \tag{9}$$

Now, to identify the correction indicated in the equation (5), we resorted to the result reported in the reference [10] where the frictional force is expressed in proportion to the power of the indexed Reynolds number, a result that allows us to reproduce the Blasius experimental results, and that cannot be obtained through the classic Navier-Stokes equations. Taking into account this result and place it now in polar coordinates allows us to infer a new stream function (10), being (γ, β) the Blasius exponent and the spatial index respectively:

$$\psi = -\nu_c(\pi - \theta) - U_l\frac{a^2}{r}\sin\theta +$$
$$\left(\frac{1}{1+\beta\gamma}\right)\left(\frac{1}{R_\beta}\right)^\gamma U_l r\sin\theta \tag{10}$$

On the edge of the developed turbulence ($\gamma \to 0$), the new stream function opens the possibility of reconstructing radial and polar velocity components, as in (11); but also it gives us a performance for the correction term indicated in (5), $\nu_1 = \nu_c/l = \nu_c/r$.

$$u_r = \nu_c/r + U_l\left(1 - (a/r)^2\right)\cos\theta,$$
$$u_r = -U_l\left(1 + (a/r)^2\right)\sin\theta \tag{11}$$

Moreover, the pressure for the most appropriate way seems to be in agreement with the differential equation for the boundary layer flow based on its polar component, equation (7). The Euler number (E_u) is obtained, unless a dependent function of the radial distance (r), resulting from the indefinite integration, which it may be bounded by the information produced by the singularity of the stagnation point at ($\theta = \pi$). In addition, if we observe closely the contribution of the viscous and inertial origin terms, is possible to detect that the second inertial term contains a new addition to the viscous forces.

Consequently, the variation of the dimensionless pressure ($\Delta p / \rho (1/2) U_l^2 = 1/E_u$) is estimated by the expression (12), (being a the radius of the pile, U_l the free velocity, and R_{el} the Reynolds number associated with the free velocity and the pile diameter ($R_{el} = (U_l D)/\nu_c$)),

$$1/E_u = (a/r)^2 \left(1 + (a/r)^2\right)\cos 2\theta - (2/R_{el})(a/r)\left(1 + 5(a/r)^2\right)\cos\theta + B(r) \quad (12)$$

The shape of the pressure distribution around the outside of the pile is shown in figure 2 and seems a curve of cardioid type, with a singularity at its tangent which represents the stagnation point.

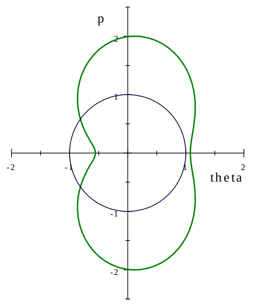

Fig. 2. Pressure variation distribution around the pile. Inward curve of the circle: positive change; external, negative.

It is noted that while in the windward side of the pile, pressures are positive, in the posterior lobe of the cardioid, downwind of the pile, pressures are negative. With the growth of the Reynolds number, this lobe decreases (downwind) until eventually it goes into the circle and becomes another stagnation point, so that the curve takes the form of a bicardiode, similar to the known pressure "inviscid" solution, which has an analogous shape of a peanut shell. However, the participation of the viscosity breaks the symmetry about the vertical axis of the bicardiode "inviscid" solution; a pressure gradient along the current is observed and therefore a pressure drags force is present, which resolves the paradox of D'Alembert. Moreover, when we plot (not shown) the pressure and the polar angle for low Reynolds number, and place away from the inviscid solution, two adjacent mounds of different heights are obtained. This great difference is a manifestation of the viscosity, and corresponds to the pressure gradient that tries to drag the obstacle (the pile in this case). When Reynolds number increases, the second mound grows, the smaller, until eventually it comes up to the first; and again, a similar configuration to the "inviscid" is obtained.

When we evaluate the pressure drag force around the cylinder, due to the symmetry, only the term with the viscosity factor contributes, so that the pressure drag force coefficient (C_p) (13) is obtained. Whereas for the frictional force coefficient C_f , (13) is obtained, and proves to be a third of the pressure.

$$C_p = 24\pi \frac{1}{R_{el}}, \quad C_f = 8\pi \frac{1}{R_{el}} \quad (13)$$

3.1 D'Alembert's paradox and Thomson theorem

Under highly turbulent flow and with the fractional Navier-Stokes equations a clear distinction between movements with high Reynolds numbers and inviscid movements is stablished, the latest governed by the Euler equations. This can be explained because with high Reynolds numbers, kinematic viscosity is not neglected, but the spatial occupation index or the relative fractal dimension becomes small, and therefore a fluid in those motion states exerts a force (similar to that on a body immersed in the fluid) that is proportional to the kinematic viscosity. Paradox is resolved and ceases to exist.

The vorticity is $\omega = rot\,\mathbf{u}$, its evolution is described by $\frac{D}{Dt}\omega = (\omega \cdot \nabla)\mathbf{u} + \nu_\alpha \nabla^\alpha \omega$; while the circulation evolution $\Gamma_C = \oint_C \mathbf{u} \cdot dl$ is given by $\frac{D\Gamma_C}{Dt} = \nu_\alpha \oint_C \nabla^\alpha \mathbf{u} \cdot dl$; so in the limit of the developed turbulence the statement of the Thomson theorem is recovered by (14):

$$\frac{D\Gamma_C}{Dt} =_{\beta \to 0} \nu_1 \oint_C \nabla^1 \mathbf{u} \cdot dl = 0 \quad (14)$$

In a turbulent fluid, vorticity is nonzero only in a certain region of the flow field, although not necessarily finite; so, the fluid can penetrate into the region of nonzero vorticity, but can't leave it. Energy dissipation in a turbulent flow occurs mainly in the region of nonzero vorticity, [6].

3.2 Dams, fan-shaped

For applications, we define the Reynolds number based on the pile diameter and the radial velocity downwind of the pile as follows: $R_e = Du_r/\nu_c$. The aspect ratio between the pile diameter and the control distance or the wake length is estimated, looking for the condition for this Reynolds number, a decrement with the radial distance. The result is as follows: on the border of inequality, the diameter must be the proportional average (geometric mean) between the control distance and the length ($2\nu_c/U_l$), which we associate with the dam as the characteristic length ($L = \nu_c/U_l$), matching the free velocity with the maximum velocity in the dam, see figure 1. Then (15) is obtained, being K the contraction factor; so, it follows that $\left(\frac{r_w}{D} > \frac{D}{2L}\right)$,

$$K\frac{r_w}{D} < \frac{D}{2L}, \quad K = 1 - \sqrt{1 - \frac{(L/D)-1}{(L/D)^2}} \qquad (15)$$

We emphasize that the estimation was achieved by means the fractional formulation, because it is impossible through inviscid approach, as the Reynolds number does not decrease with the pile downwind distance.

IV. CONCLUSIONS

The interaction between fluid and a circular pile is studied, with the fluid motion described by means of the fractional Navier-Stokes equations, and in particular by its boundary layer approximation. The pressure distribution on the edge of the cylinder, which provides the Euler number, allows us to observe the symmetry breaking caused by the viscosity on the vertical plane, meaning a drag on the cylinder in the mainstream direction. The coherence is retrieved by solving the D'Alembert's paradox and the Thomson's theorem (14), not by canceling the viscosity but by the smallness of the fractal dimension. For applications, the orientation of the turbulent wake behind the pile is given by its symmetry axis. Its control distance (15) is imposed as a level of its aspect ratio, with the pile diameter. The criterion is as follows: the pile diameter, on the border of inequality, must be located as the proportional average between the length of the turbulent wake and 2 times a characteristic length associated with the dam, whose aspect ratio, in turn, to the diameter of the pile, determines the contraction factor.

REFERENCES

[1]. B. Dargahi, The turbulent flow field around a circular cylinder. Exp. in Fluids 8, 1989, 1-12.

[2]. M. Tavellia, M. Dumbser, A staggered semi-implicit discontinuous Galerkin method for the two dimensional incompressible Navier-Stokes equations, arXiv:1407.1205v1 [math.NA] 4 Jul 2014, Preprint submitted to Applied Mathematics and Computation, July 7, 2014.

[3]. J.R. Mercado-Escalante, P A. Guido-Aldana, W. Ojeda, J. Sánchez-Sesma, E. Olvera, Saint-Venant fractional equation and hydraulic gradient. J. Math. Sys. Sci. 2(8), 2012, 494–503.

[4]. J.R. Mercado-Escalante, P.A. Guido-Aldana, J. Sánchez-Sesma, M. Íñiguez, A. González, Analysis of the Blasius' formula and the Navier-Stokes fractional equation. J. Klapp et al. (eds.), Fluid Dynamics in Physics, Engineering, and Environmental Applications, Environmental Science and Engineering (Springer-Verlag Berlin Heidelberg 2013) 475-480.

[5]. A. Sommerfeld, Mechanics of deformable bodies (Academic Press, New York, 1950).

[6]. L. D. Landau, E. M. Lifshitz, Fluid Mechanics (Pergamon Press, Oxford, 1987).

[7]. H. Rouse, Elementary Mechanics of Fluids (Dover, Publ., New York, 1946).

[8]. A. J. Chorin, J.E. Marsden, A Mathematical Introduction to Fluids Mechanics (Springer-Verlag, New York, 1992).

[9]. J.E. Marsden, M.J. Hoffman, Basic Complex Analysis (W. H. Freeman and Company, New York, 1987).

[10]. J.R. Mercado-Escalante, P.A. Guido-Aldana, J. Sánchez-Sesma, M. Íñiguez, Fórmulas para el coeficiente de arrastre y la ecuación Navier-Stokes fraccional, Tecnología y Ciencias del Agua. Vol. V, No. 2, 2014, pp. 149-160, Marzo-Abril 2014.

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